ADDITION THEORY FOR ELLIPTIC INTEGRALS

by

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INTRODUCTION

Elliptic integrals of the first, second, and third classes in Jacobi's forms are defined respectively as follows:

$$F(k, x) = \int_0^x \frac{dt}{\sqrt{(1 - t^2)(1 - k^2 t^2)}}$$

$$E(k, x) = \int_0^x \sqrt{\frac{1 - k^2 t^2}{1 - t^2}} \, dt$$

$$\pi(n, k, x) = \int_0^x \frac{dt}{(1 + n t^2)\sqrt{(1 - t^2)(1 - k^2 t^2)}}$$

where k, $0 < k < 1$, is called the modulus and n, which may be taken real or imaginary, but nonzero, is called the parameter of the integrals.

The elliptic integrals may be regarded as functions of x, defined by the aid of the above definite integrals. We are familiar with other and much simpler functions which may be defined as definite integrals. For example, we may define $\log x$ as $\int_1^x \frac{dt}{t}$, $\sin^{-1} x$ as $\int_0^x \frac{dt}{\sqrt{1 - t^2}}$, $\tan^{-1} x$ as $\int_0^x \frac{dt}{1 + t^2}$, and the theory of these functions may be based upon these definitions. For instance, the fundamental property of the logarithm is expressed by what is called the addition formula, $\log x + \log y = \log xy$. There are addition formulas for the other functions defined above, namely,

$$\sin^{-1}x + \sin^{-1}y = \sin^{-1}(x\sqrt{1 - y^2} + y\sqrt{1 - x^2})$$

$$\tan^{-1}x + \tan^{-1}y = \tan^{-1}(\frac{x + y}{1 - xy}) .$$

These formulas are usually obtained by methods involving the theory of the inverse functions of log x, $\sin^{-1}x$ and $\tan^{-1}x$, but they may be obtained without difficulty from their definitions as definite integrals. For example, let

$$\log x + \log y = \log c \qquad (1)$$

where c is to be determined in terms of x, y. Since

$\log x = \int_1^x \frac{dt}{t}$ and $\log y = \int_1^y \frac{dt}{t}$, differentiating (1),

we have

$$\frac{dx}{x} + \frac{dy}{y} = 0$$

or

$$y\,dx + x\,dy = 0. \qquad (2)$$

Integrating (2), we get

$$\int y\,dx + \int x\,dy = C_1 \qquad (3)$$

or

$$xy = C_1 \qquad (4)$$

where C_1 is an undetermined constant.

To determine C_1, let x = 1 in (4), then we have $y = C_1$; let x = 1 in (1), then log x = $\int_1^1 \frac{dt}{t}$ = 0, log y = log c, and so y = c when x = 1. Therefore C_1 = c and xy = c. Hence log x + log y = log xy. The addition formulas for $\sin^{-1}x$,

and $\tan^{-1}x$ can be derived in exactly the same way.[1]

The addition theory for elliptic integrals is of the same nature, and may be found very useful in the study of the properties of elliptic integrals. For ease and variety we will derive the addition formula for each class of elliptic integrals in a different way. For the first class, we will use a direct method of differentiation and integration due to Euler. Then we will define elliptic functions and develop the theory of the elliptic functions from the addition theory for the elliptic integrals of the first class. This will lead us to the derivation of the addition theory for the elliptic integrals of the second class. The elliptic integrals of the third class are more complex than the other two. We will employ Abel's theorem to obtain its addition formula.

<div align="center">

ADDITION THEORY OF ELLIPTIC INTEGRALS
OF THE FIRST CLASS

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Let $u_1 = \int_0^{x_1} \dfrac{dz}{\sqrt{Z}}$ and $u_2 = \int_0^{x_2} \dfrac{dz}{\sqrt{Z}}$

where $Z = (1 - z^2)(1 - k^2 z^2)$. Define $x_1 \equiv \operatorname{sn} u_1$ and $x_2 \equiv \operatorname{sn} u_2$. These definitions will be used and explained later on.

Consider a differential equation

$$\frac{dx_1}{\sqrt{X_1}} + \frac{dx_2}{\sqrt{X_2}} = 0 \qquad (1)$$

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[1] W. E. Byerly, *Integral Calculus*, p. 241.

where

$$X_1 \equiv (1 - x_1{}^2)(1 - k^2 x_1{}^2)$$

and

$$X_2 \equiv (1 - x_2{}^2)(1 - k^2 x_2{}^2).$$

Let x_1, x_2 be regarded as functions of a third variable t such that

$$\dot{x}_1 \equiv \frac{dx_1}{dt} = \sqrt{X_1}$$

and

$$\dot{x}_2 \equiv \frac{dx_2}{dt} = -\sqrt{X_2} .$$

Then

$$(\dot{x}_1)^2 = X_1 = 1 - (k^2 + 1)x_1{}^2 + k^2 x_1{}^4$$

and

$$(\dot{x}_2)^2 = X_2 = 1 - (k^2 + 1)x_2{}^2 + k^2 x_2{}^4 .$$

Differentiating and dividing by $2\dot{x}_1$ and $2\dot{x}_2$ respectively, we have

$$\ddot{x}_1 = -(k^2 + 1)x_1 + 2 k^2 x_1{}^3$$

$$\ddot{x}_2 = -(k^2 + 1)x_2 + 2 k^2 x_2{}^3 .$$

Thus

$$\ddot{x}_1 x_2 - \ddot{x}_2 x_1 = 2 k^2 (x_1{}^2 - x_2{}^2) x_1 x_2 ,$$

while

$$\dot{x}_1{}^2 x_2{}^2 - \dot{x}_2{}^2 x_1{}^2 = -(x_1{}^2 - x_2{}^2)(1 - k^2 x_1{}^2 x_2{}^2).$$

Hence

$$\frac{\ddot{x}_1 x_2 - \ddot{x}_2 x_1}{\dot{x}_1^2 x_2^2 - \dot{x}_2^2 x_1^2} = -\frac{2 k^2 x_1 x_2}{1 - k^2 x_1^2 x_2^2}$$

or

$$\frac{\ddot{x}_1 x_2 - \ddot{x}_2 x_1}{\dot{x}_1 x_2 - \dot{x}_2 x_1} = -\frac{2 k^2 x_1 x_2 (\dot{x}_1 x_2 + \dot{x}_2 x_1)}{1 - k^2 x_1^2 x_2^2}$$

that is,

$$\frac{d/dt(\dot{x}_1 x_2 - \dot{x}_2 x_1)}{\dot{x}_1 x_2 - \dot{x}_2 x_1} = -\frac{2 k^2 x_1 x_2}{1 - k^2 x_1^2 x_2^2} \frac{d}{dt}(x_1 x_2) \ .$$

Integrating both sides, we have

$$\log(\dot{x}_1 x_2 - \dot{x}_2 x_1) = \log(1 - k^2 x_1^2 x_2^2) + c$$

that is,

$$c = \log \frac{\dot{x}_1 x_2 - \dot{x}_2 x_1}{1 - k^2 x_1^2 x_2^2}$$

or

$$e^c = c_1 = \frac{x_2 \sqrt{X_1} + x_1 \sqrt{X_2}}{1 - k^2 x_1^2 x_2^2} \ .$$

Another form of the integral of (1) is obviously

$$u_1 + u_2 = \int_0^{x_1} \frac{dz}{\sqrt{Z}} + \int_0^{x_2} \frac{dz}{\sqrt{Z}} = c_2 .$$

Since both c_1 and c_2 are constant, one of them must therefore be a function of the other, say,

$$c_1 = \phi(c_2) \ .$$

Hence

$$\frac{x_2 \sqrt{X_1} + x_1 \sqrt{X_2}}{1 - k^2 x_1^2 x_2^2} = \emptyset(u_1 + u_2)$$

and the form of $\emptyset$ may be readily identified. For, since

$$u_1 = \int_0^{x_1} \frac{dz}{\sqrt{Z}}$$

and

$$u_2 = \int_0^{x_2} \frac{dz}{\sqrt{Z}} ,$$

it is clear that if $x_1 = 0$, and therefore $X_1 = 1$, we have $u_1 = 0$; and if $x_2 = 0$, and therefore $X_2 = 1$, we have $u_2 = 0$. Putting $x_2 = 0$, we have $\emptyset(u_1) = x_1 \equiv \text{sn } u_1$. Hence the form of $\emptyset$ is identified as the elliptic function sn. Thus we have

$$\text{sn}(u_1 + u_2) = \frac{x_2 \sqrt{X_1} + x_1 \sqrt{X_2}}{1 - k^2 x_1^2 x_2^2}$$

or

$$\int_0^{x_1} \frac{dz}{\sqrt{Z}} + \int_0^{x_2} \frac{dz}{\sqrt{Z}} = \int_0^{x'} \frac{dz}{\sqrt{Z}}$$

where

$$x' = \frac{x_2 \sqrt{X_1} + x_1 \sqrt{X_2}}{1 - k^2 x_1^2 x_2^2}$$

$$= \frac{x_2 \sqrt{(1 - x_1^2)(1 - k^2 x_1^2)} + x_1 \sqrt{(1 - x_2^2)(1 - k^2 x_2^2)}}{1 - k^2 x_1^2 x_2^2} .$$

This addition formula is usually called the Euler's Equation.

ELLIPTIC FUNCTIONS

If a transformation $t = \sin \theta$ is made in Jacobi's forms of the elliptic integrals, we obtain the following Legendre's forms:

$$F(k, \emptyset) = \int_0^{\emptyset} \frac{d\theta}{\sqrt{1 - k^2\sin^2\theta}} = \int_0^{\emptyset} \frac{d\theta}{\Delta\theta}$$

$$E(k, \emptyset) = \int_0^{\emptyset} \sqrt{1 - k^2\sin^2\theta}\ d\theta = \int_0^{\emptyset} \Delta\theta\ d\theta$$

$$\pi(n,k,\emptyset) = \int_0^{\emptyset} \frac{d\theta}{(1+n\ \sin^2\theta)\sqrt{1-k^2\sin^2\theta}} = \int_0^{\emptyset} \frac{d\theta}{(1+n\ \sin^2\theta)\Delta\theta}$$

where $\emptyset$ is called the amplitude of the elliptic integral and $\Delta\theta = \sqrt{1 - k^2\sin^2\theta}$ is called the delta amplitude and is always taken with the positive sign.

From Legendre's forms of the elliptic integrals, we are led to define the elliptic functions as follows:

$$u = F(k, \emptyset)$$
$$\emptyset = am\ u$$
$$x = \sin \emptyset = sn\ u$$
$$\sqrt{1 - x^2} = \cos \emptyset = cn\ u$$
$$\sqrt{1 - k^2x^2} = \Delta \emptyset = dn\ u.$$

$sn\ u$, $cn\ u$, $dn\ u$ are trigonometric functions of $\emptyset$, the amplitude of u, but they may be regarded as new and somewhat complicated

functions of u itself, and from this point of view they are
called elliptic functions of u. From the above definitions,
we can easily obtain the following fundamental formulas con-
necting the elliptic functions:

$$sn^2u + cn^2u = 1 \qquad (1)$$

$$dn^2u + k^2sn^2u = 1 \qquad (2)$$

$$\frac{d \ am \ u}{du} = dn \ u \qquad (3)$$

$$\frac{d \ sn \ u}{du} = cn \ u \ dn \ u \qquad (4)$$

$$\frac{d \ cn \ u}{du} = -sn \ u \ dn \ u \qquad (5)$$

$$\frac{d \ dn \ u}{du} = -k^2sn \ u \ cn \ u \qquad (6)$$

Among this set of formulas, (3) and (6) need explanations.
We have

$$u = \int_0^{\phi} \frac{d \ \theta}{\Delta \theta} \ .$$

Hence

$$du = \frac{d \ \phi}{\Delta \phi} = \frac{d \ am \ u}{dn \ u} \ .$$

and

$$\frac{d \ am \ u}{du} = dn \ u \ .$$

To obtain formula (6), by definition we have

$$\text{dn } u = \sqrt{1 - k^2 \text{ sn}^2 u} \ .$$

Differentiating with respect to u gives

$$\frac{d \text{ dn } u}{du} = \frac{d}{du} \sqrt{1 - k^2 \text{ sn}^2 u}$$

$$= \frac{1}{2} (1 - k^2 \text{sn}^2 u)^{-1/2}(-2k^2 \text{sn } u \text{ cn } u \text{ dn } u)$$

$$= \frac{1}{2} (\text{dn } u)^{-1}(-2 \ k^2 \text{sn } u \text{ cn } u \text{ dn } u)$$

$$= - k^2 \text{ sn } u \text{ cn } u \ .$$

In order to derive addition formulas for elliptic functions, we need to know these functions of negative u. By definition, $\triangle \phi = \sqrt{1 - k^2 \sin^2 \phi}$, we see that $\triangle(-\phi) = \triangle \phi$, and

$$\int_0^{-\phi} \frac{d \theta}{\triangle \theta} = \int_0^{\phi} \frac{d(-\theta)}{\triangle(-\theta)} = - \int_0^{\phi} \frac{d \theta}{\triangle \theta} = -u \ .$$

Thus,

$$\text{am}(-u) = -\phi = -\text{am } u.$$

Also,

$$\text{sn}(-u) = \sin(-\phi) = -\sin \phi = -\text{sn } u$$
$$\text{cn}(-u) = \cos(-\phi) = \cos \phi = \text{cn } u$$
$$\text{dn}(-u) = \triangle(-\phi) = \triangle \phi = \text{dn } u \ .$$

Therefore, we have the following set of identities:

$$am(-u) = -am\ u$$
$$sn(-u) = -sn\ u$$
$$cn(-u) = cn\ u \qquad\qquad (7)$$
$$dn(-u) = dn\ u.$$

The addition formula derived for $F(k, x)$ can be written in terms of the elliptic functions as follows:

$$sn(u_1+u_2) = \frac{sn\ u_1\ cn\ u_2\ dn\ u_2 + sn\ u_2\ cn\ u_1\ dn\ u_1}{1 - k^2\ sn^2\ u_1\ sn^2\ u_2} \qquad (8)$$

If we replace u_2 by $-u_2$ and simplify by (7), we have

$$sn(u_1-u_2) = \frac{sn\ u_1\ cn\ u_2\ dn\ u_2 - sn\ u_2\ cn\ u_1\ dn\ u_1}{1 - k^2\ sn^2\ u_1\ sn^2\ u_2} \qquad (9)$$

The addition formulas for cn and dn can be obtained[1] and are

$$cn(u_1\pm u_2) = \frac{cn\ u_1\ cn\ u_2 \mp sn\ u_1\ sn\ u_2\ dn\ u_1\ dn\ u_2}{1 - k^2\ sn^2\ u_1\ sn^2\ u_2} \qquad (10)$$

$$dn(u_1\pm u_2) = \frac{dn\ u_1\ dn\ u_2 \mp k^2\ sn\ u_1\ sn\ u_2\ cn\ u_1\ cn\ u_2}{1 - k^2\ sn^2\ u_1\ sn^2\ u_2} \qquad (11)$$

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[1] A. G. Greenhill, The Applications of Elliptic Functions, p. 112.

ADDITION THEORY FOR ELLIPTIC INTEGRALS
OF THE SECOND CLASS

The elliptic integral of the second class, $E(k, \emptyset)$, can be expressed in terms of elliptic functions, and we will employ its new form in deriving its addition theory.

We have

$$E(k, \emptyset) = \int_0^\emptyset \Delta\theta \; d\theta.$$

Let $u = F(k, \emptyset)$. Then $\emptyset = \text{am } u$ and $E(k, \emptyset)$ may be written as $E(k, \text{am } u)$ or just simply $E(\text{am } u)$. Then

$$E(\text{am } u) = \int_0^{\text{am } u} \text{dn } u \cdot d \text{ am } u.$$

But as obtained in the last section,

$$d \text{ am } u = \text{dn } u \cdot du$$

so

$$E(\text{am } u) = \int_0^u \text{dn}^2 u \cdot du.$$

Now

$$E(\text{am } u) + E(\text{am } v) = \int_0^u \text{dn}^2 u \; du + \int_0^v \text{dn}^2 v \; dv$$

$$= \int_0^u \text{dn}^2 z \; dz + \int_0^v \text{dn}^2 z \; dz$$

$$= \int_0^{u+v} \text{dn}^2 z \; dz - \int_u^{u+v} \text{dn}^2 z \; dz$$

$$+ \int_0^v \text{dn}^2 z \; dz$$

$$= E\left[am(u+v)\right] + \int_0^v dn^2 z \, dz - \int_u^{u+v} dn^2 z \, dz.$$

Replacing z by u + z in the second integral, and considering u and v as given constants, we get

$$\int_u^{u+v} dn^2 z \, dz = \int_0^v dn^2 (u + z) dz$$

and

$$E(am\ u) + E(am\ v) = E\left[am(u + v)\right] - \int_0^v \left[dn^2(u + z) - dn^2 z\right]dz.$$

From the addition formulas for dn shown in the last section, we have

$$dn(u_1 + u_2) + dn(u_1 - u_2) = \frac{2\ dn\ u_1\ dn\ u_2}{1 - k^2 sn^2\ u_1\ sn^2\ u_2}$$

and

$$dn(u_1 + u_2) - dn(u_1 - u_2) = - \frac{2\ k^2\ sn\ u_1\ sn\ u_2\ cn\ u_1\ cn\ u_2}{1 - k^2\ sn^2\ u_1\ sn^2\ u_2}.$$

Let $u_1 + u_2 = u + z$, and $u_1 - u_2 = z$. Then

$$u_1 = \frac{u}{2} + z \ , \text{ and } u_2 = \frac{u}{2}\ .$$

Hence

$$dn^2(u + z) - dn^2 z = \left[dn(u + z) + dn\ z\right]\left[dn(u + z) - dn\ z\right]$$

$$= \frac{-4k^2\ sn(\frac{u}{2} + z)cn(\frac{u}{2} + z)dn(\frac{u}{2} + z)sn\ \frac{u}{2}\ cn\ \frac{u}{2}\ dn\ \frac{u}{2}}{\left[1 - k^2\ sn^2\ \frac{u}{2}\ sn^2(\frac{u}{2} + z)\right]^2}\ .$$

Also, since

$$\frac{d}{dz}\left[1 - k^2\ sn^2\ \frac{u}{2}\ sin^2(\frac{u}{2} + z)\right]$$

$$= -2 \ k^2 \ sn^2 \ \frac{u}{2} \ sn(\frac{u}{2} + z) \, cn(\frac{u}{2} + z) \, dn(\frac{u}{2} + 2)$$

and

$$sn(u_1 + u_2) \cdot sn(u_1 - u_2) = \frac{sn^2 u_1 - sn^2 u_2}{1 - k^2 \ sn^2 u_1 \ sn^2 u_2} \, , \quad [1]$$

we get

$$\int_0^v \left[dn^2(u + z) - dn^2 z \right] dz$$

$$= -2 \ sn \frac{u}{2} \ cn \frac{u}{2} \ dn \frac{u}{2} \int_0^v \frac{2k^2 sn(\frac{u}{2} + z) \, cn(\frac{u}{2} + z) \, dn(\frac{u}{2} + z) \, dz}{\left[1 - k^2 sn^2 \frac{u}{2} \ sn^2(\frac{u}{2} + z) \right]^2}$$

$$= - \frac{2 sn \frac{u}{2} \ cn \frac{u}{2} \ dn \frac{u}{2}}{sn^2 \frac{u}{2}} \int_0^v \frac{2k^2 sn^2 \frac{u}{2} \ sn(\frac{u}{2} + z) \, cn(\frac{u}{2} + z) \, dn(\frac{u}{2} + z) \, dz}{\left[1 - k^2 sn^2 \frac{u}{2} \ sn^2(\frac{u}{2} + z) \right]^2}$$

$$= - \frac{2 sn \frac{u}{2} \ cn \frac{u}{2} \ dn \frac{u}{2}}{sn^2 \frac{u}{2}} \left[\frac{1}{1 - k^2 sn^2 \frac{u}{2} \ sn^2(\frac{u}{2} + z)} \right]_0^v$$

$$= - \frac{2 sn \frac{u}{2} \ cn \frac{u}{2} \ dn \frac{u}{2}}{sn^2 \frac{u}{2}} \left[\frac{1}{1 - k^2 sn^2 \frac{u}{2} \ sn^2(\frac{u}{2} + v)} - \frac{1}{1 - k^2 sn^4 \frac{u}{2}} \right]$$

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[1] A. G. Greenhill, The Applications of Elliptic Functions, p. 138.

$$= - \frac{2k^2 \operatorname{sn} \frac{u}{2} \operatorname{cn} \frac{u}{2} \operatorname{dn} \frac{u}{2}}{1 - k^2 \operatorname{sn}^4 \frac{u}{2}} \cdot \frac{\operatorname{sn}^2(\frac{u}{2} + v) - \operatorname{sn}^2 \frac{u}{2}}{1 - k^2 \operatorname{sn}^2 \frac{u}{2} \operatorname{sn}^2(\frac{u}{2} + v)}$$

$$= - k^2 \operatorname{sn} u \cdot \operatorname{sn} v \operatorname{sn}(u + v).$$

Therefore the required addition formula is

$$E(\operatorname{am} u) + E(\operatorname{am} v) = E\left[\operatorname{am}(u + v)\right] + k^2 \operatorname{sn} u \operatorname{sn} v \operatorname{sn}(u + v) \ .$$

In Jacobi's form, the addition formula for elliptic integrals of the second class is

$$\int_0^{x_1} \sqrt{\frac{1 - k^2 x^2}{1 - x^2}} \, dx + \int_0^{x_2} \sqrt{\frac{1 - k^2 x^2}{1 - x^2}} \, dx$$

$$= \int_0^{x'} \sqrt{\frac{1 - k^2 x^2}{1 - x^2}} \, dx + k^2 x_1 x_2 x'$$

where

$$x' = \frac{x_1 \sqrt{(1 - x_2^2)(1 - k^2 x_2^2)} + x_2 \sqrt{(1 - x_1^2)(1 - k^2 x_1^2)}}{1 - k^2 x_1^2 x_2^2} .$$

The elliptic integral of the second class arises in the determination of the length of an arc of an ellipse, and thus supplies a reason for use of the term elliptic integral. We will give an example in this connection to illustrate the application of the addition formula we have just derived.

The length of an arc of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ measured from the end of the minor axis can be found to be

$$S = \int_0^{x_1} \left(\frac{a^2 - e^2 x^2}{a^2 - x^2} \right)^{1/2} dx .$$

If we let $x = a \sin \theta$, this integral becomes

$$S = a \int_0^{\emptyset} \sqrt{1 - e^2 \sin^2 \theta} \ d\theta = aE(e, \emptyset)$$

where e, replacing k in our previous form, is the eccentricity
of the ellipse. If $x_1 = a$, $\emptyset = \pi/2$ and the length of the
elliptic quadrant is

$$S_q = a \int_0^{\pi/2} \sqrt{1 - e^2 \sin^2 \theta} \ d\theta = aE(e, \pi/2).$$

By the aid of the addition formula

$$E(am \ u) + E(am \ v) = E\left[am(u+v)\right] + k^2 sn \ u \ sn \ v \ sn(u + v),$$

it is always possible to find an arc of an ellipse differing
from the sum of two given arcs by an expression which is alge-
braic in terms of the abscissas of the extremities of the three
arcs.

Let $\emptyset = am \ u$, $\Psi = am \ v$, and $\sigma = am(u + v)$. Then the
addition formula given above becomes

$$E(k, \ \emptyset) + E(k, \Psi) = E(k, \sigma) + k^2 \sin \emptyset \sin \Psi \sin \sigma . \quad (1)$$

From the addition formulas of elliptic functions, we have

$$cn(u + v) = \frac{cn \ u \ cn \ v - sn \ u \ sn \ v \ dn \ u \ dn \ v}{1 - k^2 sn^2 \ u \ sn^2 \ v}$$

$$dn(u + v) = \frac{dn \ u \ dn \ v - k^2 \ sn \ u \ sn \ v \ cn \ u \ cn \ v}{1 - k^2 \ sn^2 \ u \ sn^2 \ v} .$$

From the second formula, we get

$$dn\ u\ dn\ v = dn(u+v)(1 - k^2 sn^2 u\ sn^2 v) + k^2 sn\ u\ sn\ v\ cn\ u\ cn\ v.$$

Substituting into the first formula and simplifying gives

$$cn(u + v) = cn\ u\ cn\ v - sn\ u\ sn\ v\ dn(u + v).$$

This indicates the relation of three angles $\emptyset$, Ψ, and σ, namely,

$$\cos \sigma = \cos \emptyset \cos \Psi - \sin \emptyset \sin \Psi \Delta\sigma \qquad (2)$$

where

$$\Delta\sigma \equiv dn(u + v) \equiv \sqrt{1 - k^2 \sin^2 \sigma}\ .$$

If we multiply (1) by a and take k equal to e, we get

$$aE(e,\ \emptyset) + aE(e,\ \Psi) = aE(e,\sigma) + \frac{e^2}{a^2} x_1 x_2 x_3 \qquad (3)$$

where x_1, x_2, and x_3 are the abscissas of the points whose amplitudes are $\emptyset$, Ψ, and σ. Let $\sigma = \pi/2$. Then $aE(e,\sigma)$ is the arc of a quadrant, and (2) reduces to

$$0 = \cos \emptyset \cos \Psi - \sin \emptyset \sin \Psi \sqrt{1 - e^2}.$$

Since $\sqrt{1 - e^2} = b/a$,

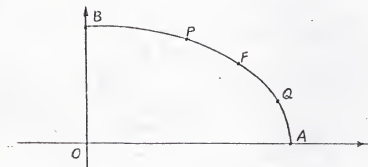$$\cos \emptyset \cos \Psi = \sin \emptyset \sin \Psi\ b/a$$

or

$$\tan \emptyset \tan \Psi = a/b \qquad (4)$$

and from (3), we get

$$aE(e, \emptyset) - \left[aE(e, \frac{\pi}{2}) - aE(e, \Psi) \right] = \frac{e^2}{a} x_1 x_2 .$$

Thus if any point P is given on the ellipse, we can get the amplitude of a second point Q from (4) and then find Q such that the arc BP minus the arc AQ, as shown in the figure, shall be equal to a quantity which is proportional to the product of the abscissas of P and Q. For a special case where P and Q

coincide at F and $\emptyset = \Psi$, we have from (4) $\tan \emptyset = \sqrt{a/b}$ and from (5)

$$\text{arc BF} - \text{arc AF} = ae^2 \sin^2 \emptyset = \frac{a^2 e^2}{a + b} = a - b.$$

This point F, which divides the quadrant into two arcs whose difference is equal to the difference between the semi-axes, is known as Fagnano's point. It has a number of curious proper-ties.[1] For example, it can be shown that the distance of the normal and tangent at Fagnano's point from the center of the ellipse are equal to a - b and $\sqrt{ab}$, respectively.

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[1]A. G. Greenhill, The Applications of Elliptic Functions, p. 182.

ADDITION THEORY FOR ELLIPTIC INTEGRALS
OF THE THIRD CLASS

The elliptic integrals of the third class denoted by $\pi(n, k, z)$, are more complicated than those in the other two classes. To derive their addition formula we will use the method suggested by Abel's Theorem.

Let C and C' be two plane curves given by the equations

$$C : F(x, y) = 0$$
$$C': \Psi(x, y) = 0.$$

Suppose that these curves have m points of intersections (x_i, y_i), where m is the product of the degrees of C and C'. Let $R(x, y)$ be a rational function of x and y, where y is a function of x as defined by $F(x, y) = 0$.

Consider the sum

$$I = \sum_{i=1}^{m} \int_{x_0, y_0}^{x_i, y_i} R(x, y) dx$$

where (x_0, y_0) is a fixed point. If some of the coefficients a_i of $\Psi(x, y)$ are regarded as continuous variables, the points (x_i, y_i) will vary and I will become a function of the variable coefficients a_i and its form can be determined. Thus for m = 3, the sum of two integrals can be expressed in terms of the third integral and I.

For determining I, the following lemma will be used.

Lemma 1. Let $F(x)$ be a polynomial of degree k and

$$G(x) = B(x - x_1)(x - x_2) \ . \ . \ . \ (x - x_n)$$

where B is a constant and the x_i are distinct constants.

By the division algorithm,

$$xF(x) = Q(x)G(x) + R(x) \qquad (1)$$

where degree of R < degree of G, or R = 0. Let $Q_0 = Q(0)$, the constant term of Q. Then

$$\sum_{i=1}^{n} \frac{F(x_i)}{G'(x_i)} = \begin{cases} 0 & \text{when } k < n - 1 \\ Q_0 & \text{when } k \geqslant n - 1 \end{cases}.$$

Proof: From (1) we have

$$\frac{xF(x)}{G(x)} = Q(x) + \frac{R(x)}{G(x)} = Q(x) + \sum_{i=1}^{n} \frac{c_i}{x - x_i}$$

where c_i is constant, and

$$G'(x) \equiv \frac{dG(x)}{dx} = B \sum_{i=1}^{n} (x - x_1) \ldots (x - x_{i-1})(x - x_{i+1}) \ldots (x - x_n)$$

$$= \sum_{i=1}^{n} \frac{G(x)}{x - x_i}.$$

Hence

$$xF(x) = Q(x)G(x) + \sum_{i=1}^{n} \frac{c_i G(x)}{x - x_i}$$

$$x_i F(x_i) = Q(x_i)G(x_i) + c_i G'(x_i) = c_i G'(x_i)$$

$$c_i = \frac{x_i F(x_i)}{G'(x_i)}.$$

Thus we have

$$\frac{xF(x)}{G(x)} = Q(x) + \sum_{i=1}^{n} \frac{x_i F(x_i)}{G'(x_i)(x - x_i)} \,. \tag{2}$$

When $x = 0$, (2) becomes

$$0 = Q(0) + \sum_{i=1}^{n} \frac{x_i F(x_i)}{G'(x_i)(0 - x_i)}$$

or

$$\sum_{i=1}^{n} \frac{F(x_i)}{G'(x_i)} = Q_0 \,.$$

Now if $k < n - 1$, $xF(x)$ is of lower degree than G, so $\dfrac{xF(x)}{G(x)}$ is a proper fraction and $Q = 0$. Hence

$$\sum_{i=1}^{n} \frac{F(x_i)}{G'(x_i)} = 0 \,.$$

The elliptic integral of third class in Jacobi's form is

$$\pi(n,k,z) = \int_{0}^{z_1} \frac{dz}{(1 + nz^2)\sqrt{(1 - z^2)(1 - k^2 z^2)}} \,.$$

Let $x = z^2$. Then the integral can be transformed into the form

$$\pi(n,k,x) = \frac{1}{2} \int_{0}^{x_1} \frac{dx}{(1 + nx)\sqrt{x(1 - x)(1 - k^2 x)}} \,. \tag{1}$$

Consider two plane curves given by

$$y^2 = x(1 - x)(1 - k^2 x) \tag{2}$$

$$y = ax + b \,. \tag{3}$$

Solving these two equations simultaneously, we get the equation

$$\Psi(x) = k^2 x^3 - (1 + k^2 + a^2) x^2 + (1 - 2ab)x - b^2 = 0 \qquad (4)$$

the roots of which give the abscissas of the points of inter-
sections of the two curves. Thus we can write

$$\Psi(x) = k^2 (x - x_1)(x - x_2)(x - x_3) = 0$$

where

$$\sum_{i=1}^{3} x_i = \frac{1}{k^2} (1 + k^2 + a^2)$$

$$\sum_{\substack{i,j=1 \\ i \neq j}}^{3} x_i x_j = \frac{1}{k^2} (1 - 2ab)$$

and

$$x_1 x_2 x_3 = \frac{b^2}{k^2} .$$

It is clear that x_1, x_2, x_3 depend upon a and b. Differentiat-
ing (4) in x_i with respect to a, we have

$$\psi'(x_i) \frac{\partial x_i}{\partial a} + \frac{\partial \Psi(x_i)}{\partial a}$$

$$= \Psi'(x_i) \frac{\partial x_i}{\partial a} + (-2ax_i^2 - 2bx_i) = \Psi'(x_i) \frac{\partial x_i}{\partial a} - 2x_i y_i = 0 .$$

Thus

$$\frac{\partial x_i}{\partial a} = \frac{2x_i y_i}{\Psi'(x_i)} .$$

Similarly, differentiating with respect to b, we have

$$\Psi'(x_i) \frac{\partial x_i}{\partial b} - 2 y_i = 0$$

so

$$\frac{\partial x_i}{\partial b} = \frac{2\, y_i}{\Psi'(x_i)} .$$

For simplicity, let

$$X = \frac{1}{(1 + nx)\sqrt{x(1 - x)(1 - k^2 x)}} .$$

Now, consider

$$I(a,\ b) = \int_0^{x_1} X\, dx + \int_0^{x_2} X\, dx + \int_{1/k^2}^{x_3} X\, dx . \qquad (5)$$

Then

$$\frac{\partial I}{\partial a} = \sum_{i=1}^{3} \frac{\partial x_i / \partial a}{(1 + nx_i)\sqrt{x_i(1 - x_i)(1 - k^2 x_i)}}$$

$$= \sum_{i=1}^{3} \frac{2\, x_i}{(1 + nx_i)\, \Psi'(x_i)} .$$

By Lemma 1, with $F(x) = x$, $G(x) = (1 + nx)\Psi(x)$,

$$\sum_{i=1}^{4} \frac{F(x_i)}{G'(x_i)} = \sum_{i=1}^{3} \frac{x_i}{(1 + nx_i)\, \Psi'(x_i)} + \frac{(-1/n)}{n\,\Psi(-1/n)} = 0$$

since $(-1/n)$ is a zero of $G(x)$. Hence

$$\frac{\partial I}{\partial a} = \frac{2/n}{n\,\Psi(-1/n)} = \frac{-2n}{k^2(1 + nx_1)(1 + nx_2)(1 + nx_3)}$$

$$= \frac{-2n}{k^2\left[1 + n\displaystyle\sum_{i=1}^{3} x_i + n^2 \displaystyle\sum_{\substack{i,j=1 \\ i \neq j}}^{3} x_i x_j + n^3 x_1 x_2 x_3\right]}$$

$$= -\frac{2n}{k^2} \left[\frac{1}{1 + \dfrac{n}{k^2}(1 + k^2 + a^2) + \dfrac{n^2}{k^2}(1 - 2ab) + \dfrac{n^3 b^2}{k^2}} \right]$$

$$= -2n \left[\frac{1}{k^2 + n(1 + k^2 + a^2) + n^2(1 - 2ab) + n^3 b^2} \right]$$

$$= \frac{-2}{p + (nb - a)^2}$$

where

$$p = 1 + n + k^2 + \frac{k^2}{n} = (1 + n)\left(1 + \frac{k^2}{n}\right) .$$

Differentiating (5) with respect to b, we have

$$\frac{\partial I}{\partial b} = \sum_{i=1}^{3} \frac{\partial x_i / \partial b}{(1 + nx_i)\sqrt{x_i(1 - x_i)(1 - k^2 x_i)}}$$

$$= \sum_{i=1}^{3} \frac{2}{(1 + nx_i)\,\Psi'(x_i)} .$$

Again by Lemma 1 with $F(x) = 1$, $G(x) = (1 + nx)\Psi(x)$,

$$\sum_{i=1}^{n} \frac{F(x_i)}{G'(x_i)} = \sum_{i=1}^{3} \frac{1}{(1 + nx_i)\Psi'(x_i)} + \frac{1}{n\,\Psi(-1/n)} = 0 .$$

Thus

$$\frac{\partial I}{\partial b} = \frac{-2}{n\,\Psi(-1/n)}$$

$$= \frac{2\,n^2}{k^2(1 + nx_1)(1 + nx_2)(1 + nx_3)}$$

$$= \cfrac{2\,n^2}{k^2\left[1 + \sum_{i=1}^{3} x_i + n^2 \sum_{\substack{i,j=1 \\ i \neq j}}^{3} x_i x_j + n^3 x_1 x_2 x_3\right]}$$

$$= \cfrac{2\,n^2}{k^2\left[1 + n(1 + k^2 + a^2) + n^2(1 - 2ab) + n^3 b^2\right]}$$

$$= \cfrac{2\,n}{p + (nb - a)^2} \; .$$

Integrating either $\partial I / \partial a$ or $\partial I / \partial b$, we have

$$I = \frac{2}{\sqrt{p}}\, \tan^{-1}\left(\frac{nb - a}{\sqrt{p}}\right) + c \quad \text{for} \quad p > 0 \tag{6}$$

$$I = - \frac{2}{nb - a} + c \qquad \text{for} \quad p = 0 \tag{7}$$

$$I = \frac{1}{\sqrt{-p}}\, \log\left[\frac{nb - a - \sqrt{-p}}{nb - a + \sqrt{-p}}\right] + c \quad \text{for } p < 0 \; . \tag{8}$$

For the determination of c, choose (a', b') with b' = 0,
such that $x_1' = 0$ and $x_3' = x_3$. Then for $p > 0$

$$I(a', b') = \int_{0}^{0} X\, dx + \int_{0}^{x_2'} X\, dx + \int_{1/k^2}^{x_3} X\, dx$$

$$= \frac{2}{\sqrt{p}}\, \tan^{-1}\left(\frac{-a'}{\sqrt{p}}\right) + c \; .$$

Thus

$$c = \int_{0}^{x_2'} X\, dx + \int_{1/k^2}^{x_3} X\, dx + \frac{2}{\sqrt{p}}\, \tan^{-1}\left(\frac{a'}{\sqrt{p}}\right) \; .$$

Substituting into (5) and (6) and using the addition formula
for $\tan^{-1} x$, we have

$$\int_0^{x_1} X\,dx + \int_0^{x_2} X\,dx = \int_0^{x_2'} X\,dx + \frac{2}{\sqrt{p}}\tan^{-1}\left[\frac{\sqrt{p}\,(nb-a+a')}{p - a'(nb-a)}\right].$$

$$(9)$$

For $p = 0$

$$c = \int_0^{x_2'} + \int_{1/k^2}^{x_3} + 2/a'.$$

Hence

$$\int_0^{x_1} X\,dx + \int_0^{x_2} X\,dx$$

$$= \int_0^{x_2'} X\,dx + \frac{2(nb - a - a')}{a'(nb - a)}.$$

$$(10)$$

For $p < 0$

$$c = \int_0^{x_2'} X\,dx + \int_{1/k^2}^{x_3} X\,dx - \frac{1}{\sqrt{-p}}\log\left[\frac{-a' - \sqrt{-p}}{-a' + \sqrt{-p}}\right].$$

Hence

$$\int_0^{x_1} X\,dx + \int_0^{x_2} X\,dx$$

$$= \int_0^{x_2'} X\,dx + \frac{1}{\sqrt{-p}}\log\left[\frac{nb - a - \sqrt{-p}}{nb - a + \sqrt{-p}} \cdot \frac{a' - \sqrt{-p}}{a' + \sqrt{-p}}\right].$$

$$(11)$$

Now we need to compute a, b and a' in terms of x, y. Remember a, b and a' are related to x, y in equations (3) and (4).

For a', b' with $b' = 0$, $x_1' = 0$, $x_3' = x_3$, we have

$$x_3 x_2' = 1/k^2$$

and

$$b = k\sqrt{x_1 x_2 x_3} = \sqrt{\frac{x_1 x_2}{x_2'}} \; . \tag{12}$$

From (3), we have

$$a' = \frac{y_3}{x_3} = \frac{y_2'}{y_2'} \tag{13}$$

$$a = \frac{y_3 - b}{x_3} = \frac{y_3}{x_3} - k^2\sqrt{x_1 x_2 x_2'} = \frac{y_2'}{x_2'} - k^2\sqrt{x_1 x_2 x_2'} \; . \tag{14}$$

Also since

$$\begin{cases} y_1 = a x_1 + b \\ y_2 = a x_2 + b \; , \end{cases}$$

we have

$$a = \frac{y_1 - y_2}{x_1 - x_2}$$

$$b = y_1 - a x_1 = y_1 - x_1\left(\frac{y_1 - y_2}{x_1 - x_2}\right) = \frac{x_1 y_2 - x_2 y_1}{x_1 - x_2}$$

$$= \frac{x_1^2 y_2^2 - x_2^2 y_1^2}{(x_1 y_2 + x_2 y_1)(x_1 - x_2)}$$

$$= \frac{x_1^2 x_2 (1 - x_2)(1 - k^2 x_2) - x_2^2 x_1 (1 - x_1)(1 - k^2 x_1)}{(x_1 y_2 + x_2 y_1)(x_1 - x_2)}$$

$$= \frac{x_1 x_2 (1 - k^2 x_1 x_2)}{x_1 y_2 + x_2 y_1} \; .$$

Hence

$$x_2' = \frac{1}{x_1 x_2}\left[\frac{x_1 y_2 + x_2 y_1}{1 - k^2 x_1 x_2}\right]^2 \tag{15}$$

Substituting the values in (12), (13), and (14), into the preceding formulas (9), (10), and (11), we have, for $p > 0$,

$$\int_0^{x_1} X\,dx + \int_0^{x_2} X\,dx$$

$$= \int_0^{x_2'} X\,dx + \frac{2}{\sqrt{p}} \tan^{-1}\sqrt{p}\left[\frac{n\sqrt{\dfrac{x_1 x_2}{x_2'}} - \dfrac{y_2'}{x_2'} + k^2\sqrt{x_1 x_2 x_2'} + \dfrac{y_2'}{x_2'}}{p - \dfrac{y_2'}{x_2'}\left(n\sqrt{\dfrac{x_1 x_2}{x_2'}} - \dfrac{y_2'}{x_2'} + k^2\sqrt{x_1 x_2 x_2'}\right)}\right]$$

$$= \int_0^{x_2'} X\,dx + \frac{2}{\sqrt{p}} \tan^{-1}\sqrt{p}\left[\frac{{x_2'}^2\left(n\sqrt{\dfrac{x_1 x_2}{x_2'}} + k^2\sqrt{x_1 x_2 x_2'}\right)}{p{x_2'}^2 + {y_2'}^2 - x_2' y_2'\left(n\sqrt{\dfrac{x_1 x_2}{x_2'}} + k^2\sqrt{x_1 x_2 x_2'}\right)}\right]$$

$$= \int_0^{x_2'} X\,dx + \frac{2}{\sqrt{p}} \tan^{-1}\sqrt{p}\left[\frac{{x_2'}^2\sqrt{\dfrac{x_1 x_2}{x_2'}}\,(n + k^2 x_2')}{\dfrac{{}^* x_2'}{n}(1 + n x_2')(n + k^2 x_2') - x_2' y_2'\sqrt{\dfrac{x_1 x_2}{x_2'}}(n + k^2 x_2')}\right]$$

$$= \int_0^{x_2'} X\,dx + \frac{2}{\sqrt{p}} \tan^{-1}\sqrt{p}\left[\frac{n x_2'\sqrt{x_1 x_2 x_2'}}{x_2'(1 + n x_2') - n y_2'\sqrt{x_1 x_2 x_2'}}\right] . \quad (16)$$

For $p = 0$

$$\int_0^{x_1} X\,dx + \int_0^{x_2} X\,dx = \int_0^{x_2'} X\,dx + \frac{2}{a'} - \frac{2}{nb - a}$$

$${}^* p{x_2'}^2 + {y_2'}^2 = (1+n)(1 + k^2/n){x_2'}^2 + x_2'(1 - x_2')(1 - k^2 x_2')$$
$$= x_2'/n(1 + n x_2')(n + k^2 x_2').$$

$$= \int_0^{x_2'} X \, dx + \frac{2x_2'}{y_2'} - \frac{2}{nx_1x_2(\frac{1-k^2x_1x_2}{x_1y_2+x_2y_1}) - \frac{y_2'}{x_2'} + k^2\sqrt{x_1x_2x_2'}} \cdot \quad (17)$$

For $p < 0$

$$\int_0^{x_1} X \, dx + \int_0^{x_2} X \, dx = \int_0^{x_2'} X \, dx + \frac{1}{\sqrt{-p}} \log\left[\frac{nb-a-\sqrt{-p}}{nb-a+\sqrt{-p}} \cdot \frac{a'-\sqrt{-p}}{a'+\sqrt{-p}}\right]$$

$$= \int_0^{x_2'} X \, dx + \frac{1}{\sqrt{-p}} \log\left[\frac{(n\sqrt{\frac{x_1x_2}{x_2'}} - \frac{y_2'}{x_2'} + k^2\sqrt{x_1x_2x_2'} - \sqrt{-p})(\frac{y_2'}{x_2'} - \sqrt{-p})}{(n\sqrt{\frac{x_1x_2}{x_2'}} - \frac{y_2'}{x_2'} + k^2\sqrt{x_1x_2x_2'} + \sqrt{-p})(\frac{y_2'}{x_2'} + \sqrt{-p})}\right]$$

$$= \int_0^{x_2'} X \, dx + \frac{1}{\sqrt{-p}} \log \frac{ny_2'\sqrt{x_1x_2x_2'} - nx_2'\sqrt{-p}\sqrt{x_1x_2x_2'} - x_2' - nx_2'^2}{ny_2'\sqrt{x_1x_2x_2'} + nx_2'\sqrt{-p}\sqrt{x_1x_2x_2'} - x_2' - nx_2'^2} \quad (18)$$

since the numerator can be simplified as

$$(n\sqrt{x_1x_2x_2'} - y_2' + k^2x_2'\sqrt{x_1x_2x_2'} - \sqrt{-p}\,x_2')(y_2' - \sqrt{-p}\,x_2')$$

$$= ny_2'\sqrt{x_1x_2x_2'} - y_2'^2 + k^2x_2'y_2'\sqrt{x_1x_2x_2'} - \sqrt{-p}\,x_2'y_2'$$

$$- nx_2'\sqrt{x_1x_2x_2'}\sqrt{-p} + \sqrt{-p}\,x_2'y_2' - k^2x_2'^2\sqrt{x_1x_2x_2'}\sqrt{-p} - x_2'^2 p$$

$$= -(px_2'^2 + y_2'^2) + y_2'\sqrt{x_1x_2x_2'}\,(n + k^2x_2') - x_2'\sqrt{-p}\sqrt{x_1x_2x_2'}\,(n+k^2x_2')$$

$$= -\frac{x_2'}{n}(1+nx_2')(n+k^2x_2') + (n+k^2x_2')(y_2'\sqrt{x_1x_2x_2'} - x_2'\sqrt{-p}\sqrt{x_1x_2x_2'})$$

$$= (n + k^2x_2')(y_2'\sqrt{x_1x_2x_2'} - x_2'\sqrt{-p}\sqrt{x_1x_2x_2'} - \frac{x_2'}{n} - x_2'^2)$$

and, similarly, the denominator is

$$(n + k^2 x_2')(y_2' \sqrt[4]{x_1 x_2 x_2'} + x_2' \sqrt{-p} \sqrt[4]{x_1 x_2 x_2'} - \frac{x_2'}{n} - x_2'^2) \ .$$

As the final step we need to change the variable x back to the original variable z. We let $x = z^2$. Then $dx = 2z\ dz$. Thus

$$\int_0^x \frac{dx}{(1+nx)\sqrt{x(1-x)(1-k^2 x)}} = \int_0^z \frac{2\ dz}{(1+nz^2)\sqrt{(1-z^2)(1-k^2 z^2)}} \ .$$

The terms outside the integral sign in formulas (16), (17), and (18) must also be changed. In particular, z_2', the transform of x_2', is

$$z_2'^2 = \frac{1}{z_1^2 z_2^2} \left[\frac{z_1^2 \sqrt[4]{z_2^2(1-z_2^2)(1-k^2 z_2^2)} + z_2^2 \sqrt[4]{z_1^2(1-z_1^2)(1-k^2 z_1^2)}}{1 - k^2 z_1^2 z_2^2} \right]^2$$

$$z_2' = \frac{z_1 \sqrt{(1 - z_2^2)(1 - k^2 z_2^2)} + z_2 \sqrt{(1 - z_1^2)(1 - k^2 z_1^2)}}{1 - k^2 z_1^2 z_2^2} \ .$$

Let $Z_2 = (1 - z_2^2)(1 - k^2 z_2^2),$

$Z_1 = (1 - z_1^2)(1 - k^2 z_1^2),$

and

$Z' = (1 - z_2'^2)(1 - k^2 z_2'^2) \ .$

Then

$$z_2' = \frac{z_1 \sqrt{Z_2} + z_2 \sqrt{Z_1}}{1 - k^2 z_1^2 z_2^2}$$

for all cases.

The terms outside the integral sign can be transformed as follows.

For $p > 0$

$$\frac{2}{\sqrt[4]{p}} \tan^{-1}\left(\frac{n\sqrt[4]{p}\ x_2'\ \sqrt{x_1 x_2 x_2'}}{x_2'(1 + nx_2') - ny_2'\ \sqrt{x_1 x_2 x_2'}} \right) =$$

$$\frac{2}{\sqrt[4]{p}} \tan^{-1}\left(\frac{n\sqrt[4]{p}\ z_1 z_2 z_2'^3}{z_2'^2(1+nz_2'^2) - n\ z_1 z_2 z_2'^2\ \sqrt{(1-z_2'^2)(1-k^2 z_2'^2)}} \right)$$

$$= \frac{2}{\sqrt{p}} \tan^{-1}\left(\frac{n\sqrt[4]{p}\ z_1 z_2 z_2'}{1 + nz_2'^2 - nz_1 z_2\ \sqrt{Z'}} \right)$$

For $p = 0$

$$\frac{2x_2'}{y_2'} - \frac{2}{nx_1 x_2\left[\dfrac{1 - k^2 x_1 x_2}{x_1 y_2 + x_2 y_1}\right] - \dfrac{y_2'}{x_2'} + k^2\sqrt{x_1 x_2 x_2'}} =$$

$$\frac{2\ z_2'^2}{z_2'\ \sqrt{(1-z_2'^2)(1-k^2 z_2'^2)}} - \frac{2}{nz_1^2 z_2^2\left[\dfrac{1 - k^2\ z_1^2 z_2^2}{z_1^2 z_2\ \sqrt{Z_2} + z_2^2 z_1\ \sqrt{Z_1}}\right]}$$

$$- \frac{\sqrt{(1-z_2'^2)(1-k^2 z_2'^2)}}{z_2'} + k^2 z_1 z_2 z_2'$$

$$= \frac{2z_2'}{\sqrt{Z'}} - \frac{2(z_1\sqrt{Z_2} + z_2\sqrt{Z_1})}{nz_1 z_2 z_2'(1-k^2 z_1^2 z_2^2)+(k^2 z_1 z_2 z_2'^2 - \sqrt{Z'})(z_1\sqrt{Z_2} + z_2\sqrt{Z_1})}$$

For $p < 0$

$$\frac{1}{\sqrt{-p}} \log\left(\frac{ny_2'\ \sqrt{x_1 x_2 x_2'} - nx_2'\ \sqrt{-p}\ \sqrt{x_1 x_2 x_2'} - x_2' - nx_2'^2}{ny_2'\ \sqrt{x_1 x_2 x_2'} + nx_2'\ \sqrt{-p}\ \sqrt{x_1 x_2 x_2'} - x_2' - nx_2'^2} \right) =$$

$$\frac{1}{\sqrt{-p}} \log \left(\frac{nz_1z_2z_2'^2 \sqrt{(1-z_2'^2)(1-k^2z_2'^2)} - nz_2'^2 \sqrt{-p}\ z_1z_2z_2' - z_2'^2 - nz_2'^4}{nz_1z_2z_2'^2 \sqrt{(1-z_2'^2)(1-k^2z_2'^2)} + nz_2'^2 \sqrt{-p}\ z_1z_2z_2' - z_2'^2 - nz_2'^4} \right)$$

$$= \frac{1}{\sqrt{-p}} \log \left(\frac{nz_1z_2 \sqrt{Z'} - n\sqrt{-p}\ z_1z_2z_2' - 1 - nz_2'^2}{nz_1z_2 \sqrt{Z'} + n\sqrt{-p}\ z_1z_2z_2' - 1 - nz_2'^2} \right)$$

Summary of addition formulas for $\pi(n, k, z)$:

$$\int_0^{z_1} \frac{dz}{(1 + nz^2)\sqrt{(1-z^2)(1-k^2z^2)}} + \int_0^{z_2} \frac{dz}{(1+nz^2)\sqrt{(1-z^2)(1-k^2z^2)}}$$

$$= \int_0^{z_2'} \frac{dz}{(1 + nz^2)\sqrt{(1 - z^2)(1 - k^2z^2)}}$$

$$+ \begin{cases} \dfrac{1}{\sqrt{p}} \tan^{-1} \dfrac{n\sqrt{p}\ z_1z_2z_2'}{1 + nz_2'^2 - nz_1z_2\sqrt{Z'}} & \text{for } p > 0 \\[3ex] \dfrac{z_2'}{\sqrt{Z'}} - \dfrac{(z_1\sqrt{Z_2} + z_2\sqrt{Z_1})}{nz_1z_2z_2'(1-k^2z_1^2z_2^2) + (k^2z_1z_2z_2'^2 - \sqrt{Z'})(z_1\sqrt{Z_2} + z_2\sqrt{Z_1})} & \\ & \text{for } p = 0 \\[3ex] \dfrac{1}{2\sqrt{-p}} \log \dfrac{nz_1z_2\sqrt{Z'} - n\sqrt{-p}\ z_1z_2z_2' - 1 - nz_2'^2}{nz_1z_2\sqrt{Z'} + n\sqrt{-p}\ z_1z_2z_2' - 1 - nz_2'^2} & \text{for } p < 0 \end{cases}$$

where

$$z_2' = \frac{z_1\sqrt{Z_2} + z_2\sqrt{Z_1}}{1 - k^2z_1^2z_2^2}$$

$$Z_1 = (1 - z_1^2)(1 - k^2\ z_1^2)$$

$$Z_2 = (1 - z_2^2)(1 - k^2\ z_2^2)$$

$$Z' = (1 - z_2'^2)(1 - k^2 z_2'^2)$$

and

$$p = 1 + n + k^2 + \frac{k^2}{n} = (1 + n)(1 + \frac{k^2}{n}) \ .$$

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ADDITION THEORY FOR ELLIPTIC INTEGRALS

by

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Elliptic integrals were so defined that they could be regarded as functions of their individual upper limits. In this paper formulas were obtained to express the sum of two elliptic integrals in terms of a third integral plus some special function of the two upper limits.

The addition theory for the elliptic integrals of the first class in Jacobi's form was derived by a method of direct differentiation and integration. After the integral had been transformed into Legendre's form, elliptic functions were defined with properties analogous to those of trigonometric functions. By the aid of the addition theory of the elliptic integrals of the first class, many addition formulas for elliptic functions could be obtained. These formulas in turn were used to develop the addition theory for the elliptic integrals of the second class.

The addition theory for the elliptic integrals of the third class was established by the aid of Abel's Theorem. The integrand was reduced to a rational function $R(x,y)$, where y was a function of x of degree three as defined by $F(x,y) = 0$. The abscissas of three intersecting points of the curve $F(x,y) = 0$ and a varying line $\Psi(x,y) = 0$ were found to depend on the variable coefficients of $\Psi(x,y)$. The sum I of three integrals with these abscissas as their upper limits became a function of the coefficients of $\Psi(x,y)$, and its form could be determined. Thus the sum of two integrals could be expressed in terms of the third integral and I.

Applications of the derived addition theory were suggested to further the study of the properties of elliptic integrals.